ON A SECOND-ORDER DIFFERENTIAL GAME

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Optimal feedback controls are determined in a second-order differential game of guidance. The problem to be investigated does not fall into the class of differential games for which methods for constructing the optimal controls are known at the present time.

1. Consider the controlled system

$$dx_1/dt = -\lambda_1 x_1 + u + p_1 + v \quad (\lambda_2 > \lambda_1 > 0) \tag{1.1}$$
$$dx_2/dt = -\lambda_2 x_2 + ku + p_2 + lv \quad (k > 0,\ l < 0)$$

Here x_1, x_2 are components of a two-dimensional phase vector x; p_1, p_2 are arbitrary numbers; u and v are, respectively, the controls of the first and second players, subject to the constraints

$$|u(t)| \leqslant \mu, \quad \mu > 0, \qquad |v(t)| \leqslant \nu, \quad \nu > 0 \tag{1.2}$$

Our aim is to find a method for the first (second) player to behave by a feedback rule so as to guarantee him the smallest (largest) time for taking system (1.1) from an arbitrary position x_0 in plane X to the origin for any behavior of the second (first) player.

By the term "the realization $u(\cdot)$ $(v(\cdot))$" we agree to mean a measurable time function $u(t)$ $(v(t))$, $t_0 \leqslant t < \infty$, satisfying constraint (1.2) for any t and stimulated by the first (second) player during the game by some method. By the term "the program $u(\cdot)$ $(v(\cdot))$" we shall mean a measurable time function $u(t)$ $(v(t))$ satisfying constraint (1.2) for any t and specified a priori on the interval $t_0 \leqslant t < \infty$. We formulate the problem from the first player's viewpoint. We take it that for $t \geqslant t_0$ the first player can collide with any realization $v(\cdot)$. The first player is obliged to construct his own control by a feedback rule in discrete form with the aid of the functions $u[x]$, $\delta[x]$. The discrete time step $\delta[x] > 0$ defines the size of the semi-interval $t^* \leqslant t < t^* + \delta[x[t^*]]$ during which the control u is held constant. It depends on the position $x[t^*]$, where it is chosen in accordance with the function $u[x]$.

The functions $u[x]$, $\delta[x]$ are called admissible if when the situation arises that the switching instant of control u tends to the limit t_* from the left, not coinciding with the instant the phase point hits on the origin, the solution of the system (1.1) can be prolonged for $t \geqslant t_*$ and if the number of such instants t_* cannot be infinite on any finite interval. The pair consisting of the function $u[x]$ and of the sequence $(\delta_n[x])$ is called a tactic of the first player and denoted $\{u, \delta\}$. We say that a tactic $\{u, \delta\}$ is admissible if the functions $u[x]$, $\delta_n[x]$ are admissible for any n.

Problem 1. Find the optimal admissible tactic $\{u^\circ, \delta^\circ\}$ for which the inequality

$$T_u[x_0] = \lim_{n\to\infty} \sup\nolimits_{v(\cdot)} T[x_0; u^\circ, \delta_n^\circ, v(\cdot)] \leqslant \sup\nolimits_{v(\cdot)} T[x_0; u, \delta, v(\cdot)]$$

is fulfilled for any initial position x_0 and any admissible functions $u[x]$, $\delta[x]$. Here $T[x_0; u, \delta, v(\cdot)]$ is the transition time of system (1.1) from the point x_0 to the origin under the functions $u[x]$, $\delta[x]$ and the realization $v(\cdot)$. The least upper bound is taken over all possible realizations $v(\cdot)$. The bar over the limit sign signifies the least upper bound.

Let us now formulate the problem from the second player's viewpoint. We take it that for $t \geqslant t_0$ the second player can collide with any realization $u(\cdot)$. The second player is obliged to construct his own control by a feedback rule in discrete form with the aid of the function $v[x]$ and of the discrete time step $\Delta > 0$. The discrete step Δ does not depend on x and defines the size of the semi-interval $t^* \leqslant t < t^* + \Delta$ during which the control v is held constant. It depends on the position $x[t^*]$, where it is chosen in accordance with the function $v[x]$. We fix an arbitrary decreasing sequence (Δ_n) converging to zero.

Problem 2. Find an optimal function $v^\circ[x]$ for which the inequality

$$T_v[x_0] = \underline{\lim}_{n\to\infty} \inf_{u(\cdot)} T[x_0; v^\circ, \Delta_n, u(\cdot)] \geqslant \inf_{u(\cdot)} T[x_0; v, \Delta, u(\cdot)]$$

is fulfilled for any initial position x_0 and any $v[x]$, Δ. Here $T[x_0; v, \Delta, u(\cdot)]$ is the transition time of system (1.1) from the point x_0 to the origin under the function $v[x]$, the discrete step Δ, and the realization $u(\cdot)$. The greatest lower bound is taken over all possible realizations $u(\cdot)$. The bar under the limit sign signifies the greatest lower bound. We note the obvious inequality

$$T_v[x_0] \leqslant T_u[x_0], \qquad x_0 \in X \tag{1.3}$$

2. We stipulate certain notation and definitions and we state the conditions (Lemmas 2.1 and 2.2, without proof) under which the time $T_u[x_0] = \infty$. By the letter V we denote the segment consisting of points x with the coordinates $x_1 = p_1 + v$, $x_2 = p_2 + lv$, $|v| \leqslant \nu$. We denote the origin by m.

Lemma 2.1. If the segment V is intersected by the straight line $x_2 = kx_1$, then the time $T_v[x_0] = T_u[x_0] = \infty$ for any initial position $x_0 \neq m$.

Taking this result into account, in what follows we accept that the segment V lies strictly to one side of the straight line $x_2 = kx_1$. To be specific we assume that it is located strictly to the left of it.

The phase portrait of system (1.1) with $u = \mathrm{const}$, $v = \mathrm{const}$ is a stable node with an equilibrium position at the point

$$h(u, v) = \begin{pmatrix} \lambda_1^{-1}(u + p_1 + v) \\ \lambda_2^{-1}(ku + p_2 + lv) \end{pmatrix}$$

We denote the points $h(-\mu, \nu)$, $h(\mu, \nu)$, $h(\mu, -\nu)$, $h(-\mu, -\nu)$ by h_1, h_2, h_3, h_4, respectively (Fig. 1). We say that system (1.1) is attracted to the point $h(u^*, v^*)$ at instant t if at this instant the velocity vector $\dot{x}(t)$ coincides with the velocity vector of system (1.1) as it moves from the point $x(t)$ by virtue of $u = u^*$, $v = v^*$. Therefore, if the values $u(t)$ and $v(t)$ satisfying constraint (1.2) are realized at instant t, then at this instant system (1.1) is attracted to a point $h(u(t), v(t))$ belonging to the parallelogram $h_1h_2h_3h_4$. The union over v, $|v| \leqslant \nu$, of parallel straight lines

$$k\lambda_1 x_1 - \lambda_2 x_2 + p_2 + lv - kp_1 - kv = 0 \tag{2.1}$$

each of which passes through the points $h(-\mu, v)$ and $h(\mu, v)$, is called the strip V. By M_1N_1 (M_2N_2) we denote the lower (upper) boundary line of the strip. The part of plane X lying strictly below the line M_1N_1 we denote by X_1. The assertion:

the point m belongs to set X_1, is equivalent to the assertion: the segment V lies strictly to the left of the straight line $x_2 = kx_1$. Through the points h_1, h_3 and h_2, h_4 we draw, respectively, the straight lines P_1R_1 and P_2R_2. By means of these straight lines we delineate a set E in X_1(Fig. 1). We include the intersection of the straight line P_1R_1 (P_2R_2) with X_1 in E if the angular coefficient of this straight line is finite and negative (positive); otherwise, we do not include this intersection in E.

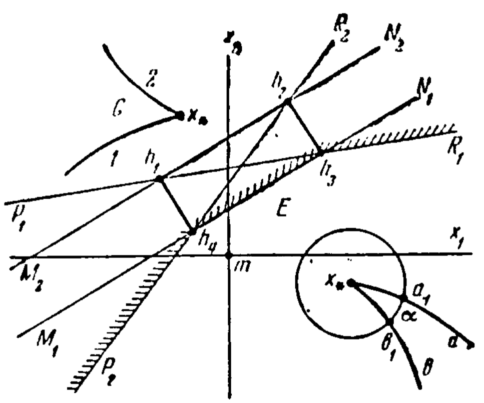

Fig. 1

Lemma 2.2. Let $m \in X_1 \setminus E$. Then the time $T_v [x_0] = \infty$ for any initial position $x_0 \neq m$. The time $T_v [x_0] = \infty$ for any $x_0 \neq m$ excepting initial positions on the positive part of the x_1-axis and on the negative part of the x_2-axis. The time $T_v[x_0] < \infty$ for initial positions on the positive (negative) part of the x_1-axis (x_2 -axis) if the straight line P_1R_1 (P_2R_2) coincides with the x_1-axis (x_2-axis); otherwise, $T_v [x_0] = \infty$.

Let x_*a and x_*b be two smooth nonselfintersecting curves issuing from the one point x_* and not coinciding within some neighborhood of this point (Fig. 1). Let us assume also that the angle between the vectors tangent to the curves x_*a and x_*b at the point x_* is different from π; we accept that each tangent vector is directed to the side of the motion along its own curve from the point x_*. Under the conditions listed there exists a number $\varepsilon_0 > 0$ such that for any positive $\varepsilon \leqslant \varepsilon_0$ the circle of radius ε with center at point x_* is intersected by each of the curves x_*a and x_*b at only one point and the angle α subtended at the vertex x_* by the arc of the circle between the points of intersection is different from 0 and π. We denote the point of intersection of the circle with the curve x_*a (x_*b) by a_1 (b_1). We accept that the angle α is counted clockwise from point a_1. We say that the relation $x_*a < x_*b$ $(x_*a > x_*b)$ exists between the curves x_*a and x_*b if for at least one $\varepsilon \leqslant \varepsilon_0$ the angle α is less (greater) than π. In this case we say that the curve x_*a is a left (right) curve relative to the curve x_*b.

Let us introduce the concept of extreme curves. Consider the rectangle Π_1 (Π_2) whose diagonal is the segment $[h_2, h_3]$ $([h_1, h_4])$ and whose sides are parallel to the coordinate axes. The intersection of set Π_1 (Π_2) with the interior of strip V is denoted by F_1 (F_2). Let x_* be any point of the plane, not belonging to F_1 (F_2). We construct a set G_1 (x_*) $(G_2$ $(x_*))$ consisting of all points of the plane to which we can take the system

$$\begin{aligned} dx_1/d\tau &= \lambda_1 x_1 - u - p_1 - v \qquad (|u(\tau)| \leqslant \mu, \ \ |v(\tau)| \leqslant \nu) \\ dx_2/d\tau &= \lambda_2 x_2 - ku - p_2 - lv \end{aligned} \tag{2.2}$$

from x_* by means of a program $v(\tau)$, $\tau_0 \leqslant \tau < \infty$, with $u = \mu$ $(u = -\mu)$. System (2.2) corresponds to system (1.1) in reverse time $\tau = -t$. The closure $\overline{G}_1$ (x_*) $(\overline{G}_2(x_*))$ of set $G_1(x_*)$ $(G_2$ $(x_*))$ is a curvilinear cone with vertex at point x_* (i.e. the set bounded by the two curves issuing from the one point x_* and not intersecting away from this point) with smooth boundaries and an angle at the vertex not equal to π. That boundary of cone $\vec{G}_1(x_*)$ $(\vec{G}_2(x_*))$ which is the left curve relative to the second boundary is called an extreme curve $r^{(1)}(x_*, \mu)$ $(r^{(1)}(x_*, -\mu))$; the second boundary of the cone is called the extreme curve $r^{(2)}(x_*, \mu)$ $(r^{(2)}(x_*, -\mu))$. The cone $\vec{G}_1(x_*)$ for some point x_* has been constructed on Fig. 1. It is denoted by G, the digit *1* denotes the curve $r^{(1)}(x_*, \mu)$, the digit *2* denotes the curve $r^{(2)}(x_*, \mu)$. By $[ab)$ we denote the set of all points of the arc ab, including point a but not including b. We introduce the sets (ab), $[ab]$, $(ab]$ similarly.

3. In Sects. 3, 4 we consider that the set E contains point m within it. The relation $r^{(1)}(m, \mu) > r^{(2)}(m, -\mu)$ is valid under such a condition. Let us give a meaningful description of a certain set A. It will be clear subsequently that it is a maximal set for any point x_0 whose time $T_u[x_0] < \infty$. All possible locations of the segments $[h_1, h_4]$ and $[h_2, h_3]$ (under the condition $m \in E$) fall into three Groups (equivalent definitions are given within parentheses) :

1. The segment $[h_1, h_4]$ lies above, but not necessarily strictly, the x_1-axis (the curves $r^{(1)}(m, \mu)$, $r^{(2)}(m, -\mu)$ do not intersect strip V).

2. The segment $[h_1, h_4]$ lies strictly below the x_1-axis while the point h_3 does not (the curve $r^{(1)}(m, \mu)$ does not intersect strip V while the curve $r^{(2)}(m, -\mu)$ does).

3. The segment $[h_1, h_4]$ and the point h_3 lie strictly below the x_1-axis (the curve $r^{(1)}(m, \mu)$ intersects strip V). We separate the first Group into two cases : 1.1. The curves $r^{(1)}(m, \mu)$ and $r^{(2)}(m, -\mu)$ intersect away from point m; 1.2. The curves $r^{(1)}(m, \mu)$ and $r^{(2)}(m, -\mu)$ do not intersect away from point m.

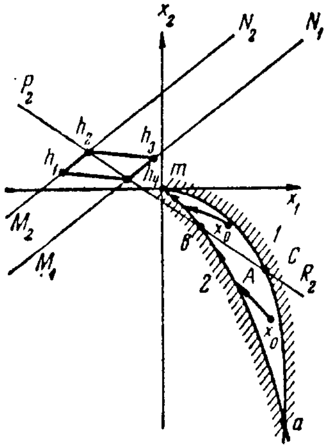

Fig. 2.

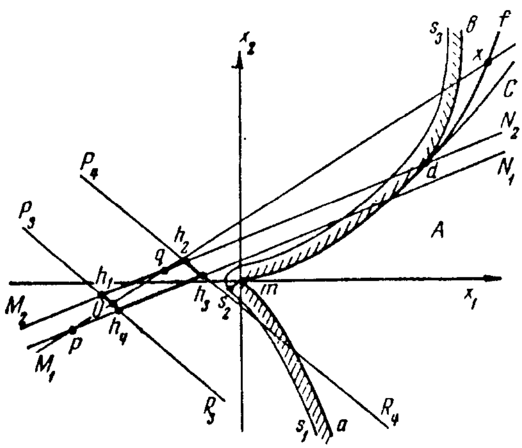

Fig. 3.

Case 1.1 is possible only if both curves $r^{(1)}(m, \mu)$ and $r^{(2)}(m, -\mu)$ pass into the fourth quadrant and are intersected away from point m by the straight line P_2R_2; we denote the points of intersection, respectively, by b and c. The point of intersection, other than m, of the curves $r^{(1)}(m, \mu)$, and $r^{(2)}(m, -\mu)$ is unique and lies below the straight line P_2R_2; we denote it by a. The set A is a closed set bounded by the curve $mcabm$. A typical construction of it is shown in Fig. 2. The digits *1*, *2* denote the curves $r^{(2)}(m, -\mu)$, $r^{(1)}(m, \mu)$, respectively.

Case 1.2. As A we take the closed curvilinear cone bounded by the curves $r^{(1)}(m, \mu)$ and $r^{(2)}(m, -\mu)$.

We turn to the second Group. First of all we note that the curve $r^{(2)}(m, -\mu)$ intersects the line M_2N_2; let d be the point of intersection. We separate the second Group into three cases: 2.1. The point d lies on the line M_2N_2 to the right of point h_2; 2.2. The points d and h_2 coincide; 2.3. The point d lies on the line M_2N_2 to the left of point h_2.

In cases 2.1. (Fig 3) and 2.2 we construct the curve $r^{(2)}(d, \mu)$ and denote it db. By ma (dc) we denote the curve $r^{(1)}(m, \mu)$ $(r^{(2)}(d, -\mu))$. Let A_1 (A_2) be the closed curvilinear cone bounded by the curves mdc and ma (db and dc). We set $A = A_1 \cup A_2$.

The peculiarity of case 2.3 is that the segments $[h_1, h_4]$, $[h_2, h_3]$ lie strictly on different sides relative to the union of the curves $r^{(1)}(m, \mu)$ and $r^{(2)}(m, -\mu)$. In this case $A = X$.

From the definition of the third Group it follows that both curves $r^{(1)}(m, \mu)$ and $r^{(2)}(m, -\mu)$ intersect strip V. We denote by e (d) the point of intersection of the curve $r^{(1)}(m, \mu)$ $(r^{(2)}(m, -\mu))$ with the line M_1N_1 (M_2N_2). We construct the curves $r^{(1)}(e, -\mu)$ and $r^{(2)}(d, \mu)$. We separate the third Group into four cases:

3.1. The curves $r^{(1)}(m, \mu)$ and $r^{(2)}(m, -\mu)$ intersect in the set $\bar{X}_1$ away from point m.

3.2. The curves $r^{(2)}(m, -\mu)$ and $r^{(1)}(e, \mu)$ intersect at the limits of strip V away from point e.

3.3. The curves $r^{(2)}(d, \mu)$ and $r^{(1)}(e, -\mu)$ intersect away from point d.

3.4. The curves $r^{(1)}(m, \mu)$ and $r^{(2)}(m, -\mu)$ do not intersect in the set $\bar{X}_1$ away from point m the curves $r^{(2)}(m, -\mu)$ and $r^{(1)}(e, \mu)$ do not intersect at the limits of strip V and the curves $r^{(2)}(d, \mu)$ and $r^{(1)}(e, -\mu)$ do not intersect.

Case 3.1 is possible only if the curves $r^{(1)}(m, \mu)$ and $r^{(2)}(m, -\mu)$ are intersected away from point m by the straight line P_1R_1; we denote the points of intersection by b and c, respectively. The point of intersection, other than m of the curves $r^{(1)}(m, \mu)$, and $r^{(2)}(m, -\mu)$ is unique and lies above the straight line P_1R_1, we denote it by a. The set A is the closed set bounded by the curve $mcabm$.

In case 3.2, (3.3) a denotes the point of intersection of the curves $r^{(2)}(m, -\mu)$ and $r^{(1)}(e, \mu)$ $(r^{(2)}(d, \mu)$ and $r^{(1)}(e, -\mu))$. The set A is the closed set bounded by the curve $maem$ $(mdaem)$. The set A for case 3.3 is shown in Fig. 4.

Case 3.4. By A_1, A_2, A_3, respectively, we denote the closed curvilinear cones whose boundaries are the curves $r^{(2)}(m, -\mu)$ and $r^{(1)}(m, \mu)$, $r^{(2)}(d, \mu)$ and $r^{(2)}(d, -\mu)$, $r^{(1)}(e, \mu)$ and $r^{(1)}(e, -\mu)$. The set $A = A_1 \cup A_2 \cup A_3$.

The following assertion is valid for any of the cases described.

Lemma 3.1. The time $T_v[x_0] = T_u[x_0] = \infty$ for any initial position $x_0 \in B = X \setminus A$

The lemma is proved by constructing the optimal function $v^\circ[x]$, $x \in B$, which guarantees to the second player the time $T_v[x_0] = \infty$ for any initial position $x_0 \in B$. That the time $T_u[x_0]$, $x_0 \in B$, is infinite follows here from inequality (1.3).

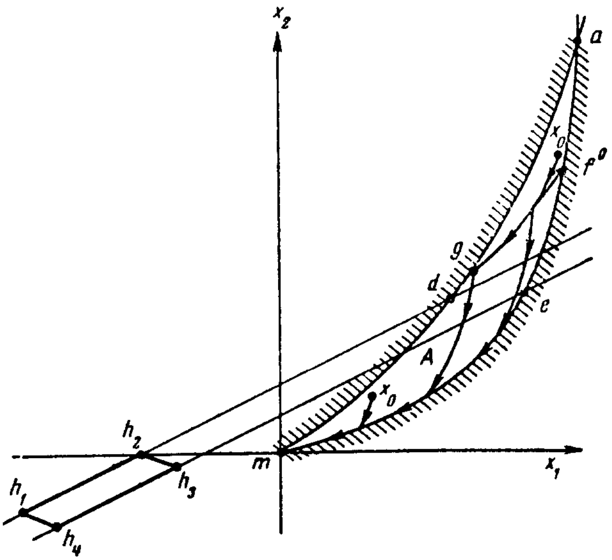

Fig. 4

The construction of the function $v^\circ[x]$, $x \in B$, is different for different cases. As an example we present the construction for case 2.1. In case 2.1. the boundary of set A consists of two smooth curves ma and mdb (Fig. 3). Through the points h_2, h_3 we draw a straight line P_4R_4. The part of the curve ma (mdb), located above the straight line P_4R_4 (M_2N_2), is a trajectory of system (1.1) for $u = \mu$, $v = -\nu$ ($v = \nu$), while the part located below this straight line is a trajectory of system (1.1) for $u = \mu$ ($u = -\mu$), $v = \nu$.

We consider a neighborhood O of set A of sufficiently small radius such that the segments $[h_1, h_4]$ and $[h_1, h_2]$ lie outside this neighborhood. The intersection of the boundary of neighborhood O with the trajectory of system (1.1) from point m for $u = -\mu$, $v = -\nu$ is denoted by s_2. We denote the boundary of neighborhood O by $s_1s_2s_3$ (Fig. 3). The curve $(ms_2]$ is the part of the trajectory mentioned above, passing into the third quadrant outside set A and outside strip V. The set $O \setminus A$ divides this curve into two parts. Let F_1 be that one of them that is adjacent to the curve ma and let $F_2 = (O \setminus A) \setminus F_1$. We include the curve ms_2 in F_2. If the straight line P_4R_4 intersects set F_1, we denote by $F_1^{(1)}$ its subset lying below this straight line and by $F_1^{(2)}$ the complement of $F_1^{(1)}$ in F_1. If there is no intersection, we set $F_1^{(1)} = F_1$.

It is not difficult to prove the validity of the following assertion. 1) If $x_0 = x(t_0) \in (ms_2]$ and $v = \nu$ ($v = -\nu$), then there exists a number $\Delta t > 0$, not dependent on x_0, such that on the interval $[t_0, t_0 + \Delta t)$ the system (1.1) will be moved outside the set $F_1 \cup A$ for any realization $u(\cdot)$. In set B we define a function $v^\circ[x]$ in the following way. In the set $F_1 \cup F_2 \subset B$ we put

$$v^\circ[x] = \begin{cases} \nu, & \text{if } x \in F_2 \cup F_1^{(1)} \\ -\nu, & \text{if } x \in F_1^{(2)} \end{cases}$$

The function $v^\circ[x]$ is given arbitrarily in the set $B \setminus (F_1 \cup F_2)$ Let us assume that at some instant t system (1.1) finds itself in the set $F_2 \cup F_1^{(1)}$ ($F_1^{(2)}$) and that $v(t) = v^\circ[x(t)]$. Then for any value $u(t)$ it is attracted at this instant to one of the points of the segment $[h_1, h_2]$ ($[h_3, h_4]$). Relying on this property and taking into account the above-described nature of the boundary curves of set A, we can prove the following assertion. 2) Suppose that the second player applies a discrete scheme on the basis of the function $v^\circ[x]$ and that $x_0 \in F_1$ ($x_0 \in F_2$). Then for a step of the discrete scheme $\Delta_n \leqslant \Delta(x_0)$, where $\Delta(x_0) > 0$ is a sufficiently small number, the system (1.1), for any realization $u(\cdot)$ goes into the curve $(s_1 s_2 m)$ ($[s_2\ s_3)$) in a finite time, without hitting on the boundary of set A up to the instant of going into the curve mentioned.

From assertions 1), 2) it follows that if the second player applies a discrete scheme on the basis of the function $v^\circ[x]$, then from any initial position $x_0 \in F_1 \cup F_2$ with a sufficiently small step of the discrete scheme, system (1.1) is carried out to the curve $s_1\ s_2\ s_3$, without hitting on the boundary of set A upto the instant of going into the curve. On the basis of this conclusion we can show that for any $x_0 \in B$, with a sufficiently small step of the discrete scheme, it is impossible for system (1.1) to fall into the boundary of set A (and, hence, also into A) for $t \geqslant t_0$ The latter proves the optimality of the function $v^\circ[x]$, $x \in B$.

4. Let us solve Problems 1, 2 for initial positions x_0 in the set A. We examine case 2.1. Consider the family L of all possible trajectories of system (2.2) with $v = \nu$, starting at point d. Any trajectory of this family passes into set A_2. We select an arbitrary trajectory from L and we denote it df (Fig. 3). Let C be a maximal closed subset of set A, located to the right of the curve mdf, and let $D = A \setminus C$. We state an auxiliary rule for forming the realizations $u(\cdot)$ in the set $A \setminus \{m\}$ by the feedback principle.

Rule 1. The value $u(t)$ at instant t equals $-\mu$ (μ), if $x(t) \in C \setminus \setminus([ma) \cup (df))$ ($x(t) \in D$). If $x(t) \in (df)$ the value $u(t)$ is chosen in accordance with the value $v(t)$ from the condition of moving along this curve in the direction of point d; if, however, such a choice is not possible, we set $u(t) = -\mu$. If $x(t) \in \in (ma)$, $u(t)$ is chosen in accordance with $v(t)$ from the condition of moving along this curve in the direction of point m.

Let us explain the choice of the value $u(t)$ on the curves (df) and (ma). From the definition of the curve df, it follows that system (1.1), moving along this curve toward point d when $v = \nu$, is attracted at each instant t to some point $q(t)$ of segment $[h_1, h_2]$. We draw a straight line through the points $x(t)$ and $q(t)$ The segment $[p(t), q(t)]$ – the intersection of this straight line with strip V – does not necessarily belong wholly to the parallelogram $h_1 h_2 h_3 h_4$. Namely, a part of this segment – the interval $[p(t), o(t))$ ($o(t)$ is the point of intersection of the segments $[h_1, h_4]$ and $[p(t), q(t)]$) – can lie to the left of the segment $[h_1, h_4]$ (Fig. 3). The interval $p(t), o(t))$ also picks out those values $v(t)$ (see (2.1)) from each of which it is impossible to choose a value $u(t)$, satisfying constraint (1.2) and directing the vector $x'(t)$ along the tangent to the curve df on the side of point d. For such values $v(t)$ in Rule 1 we have set $u(t) = -\mu$, and, by the same token, the vector $x^\cdot(t)$ is directed into the interior of set C. For values $v(t)$, corresponding to points of the segment $[o(t), q(t)]$, the vector $x^\cdot(t)$ can be directed along the tangent to the curve df on the side of point d. The vector $x^\cdot(t)$ has least length when $v(t) = \nu$. At any point $x(t) \in (ma)$ from any $v(t)$ we can choose $u(t)$, directing the vector $x^\cdot(t)$ along the tangent to ma on the side of point m. The tangent vector $x^\cdot(t)$ has least length when $v(t) = -\nu$.

For any realization $v(\cdot)$ Rule 1 allows us to transfer system (1.1) from any point $x_0 \in A$ to the point m, in finite time, without its leaving set A. (Recall that by a realization $v(\cdot)$ we mean a time function $v(t)$, $t_0 \leqslant t < \infty$, stimulated by the second player during the game and not necessarily specified a priori.)

We denote the transition time by $T^{(1)}[x_0, v(\cdot)]$. By $T^{(1)}[x_0]$ we denote its least upper bound over all possible realizations $v(\cdot)$.

We introduce the set $U[x_0, v(\cdot)\,|\,A]$ $(U[x_0, v(\cdot)\,|\,D])$ of programs $u(\cdot)$: a program $u(\cdot)$ belongs to set $U[x_0, v(\cdot)\,|A](U[x_0, v(\cdot)|D])$ if in moving from the point $x_0 \in A$ $(x_0 \in D)$ by virtue of programs $v(\cdot)$ and $u(\cdot)$ system (1.1) hits point m (curve df) in the finite time $T_1[x_0; u(\cdot), v(\cdot)]$ $(T_2[x_0; u(\cdot), v(\cdot)])$ without leaving set A (D) upto the instant of the hit. From the set of all possible programs $v(\cdot)$ we pick out a set $V[x_0|C]$: a program $v(\cdot)$ belongs to set $V[x_0|C]$ if in moving from the point $x_0 \in C$ by virtue of this program with $u = -\mu$, system (1.1) does not leave set C upto the instant of hitting onto curve ma. We pose an auxiliary problem.

P r o b l e m 4.1. Find right-piecewise-continuous programs $v^*(\cdot) \in V[x_0|C]$, $u^*(\cdot) \in U[x_0, v(\cdot)\,|A]$, satisfying the relation

$$T_1[x_0] = \max_{v(\cdot)} \min_{u(\cdot)} T_1[x_0; u(\cdot), v(\cdot)] = T_1[x_0; u^*(\cdot), v^*(\cdot)], \qquad x_0 \in C$$

where the maximum is taken over all programs $v(\cdot) \in V[x_0|C]$ and the minimum over all programs $u(\cdot) \in U[x_0, v(\cdot)\,|\,A]$.

L e m m a 4.1. The solution of Problem 4.1 is unique for any initial position $x_0 \in C$ For any $x_0 \in C$ we have $v^*(t) \equiv -\nu$, and the program $u^*(\cdot)$ coincides with the realization $u(\cdot)$, formed in accordance with Rule 1 with $v(t) \equiv -\nu$. The program $v^*(\cdot)$ satisfies the relation

$$T^{(1)}[x_0, v^*(\cdot)] = \max_{v(\cdot)} T^{(1)}[x_0, v(\cdot)], \quad x_0 \in C \tag{4.1}$$

where the maximum is taken over all programs $v(\cdot) \in V[x_0|C]$.

P r o o f. The motion of system (1.1) by virtue of the programs $v^*(\cdot)$ and $u^*(\cdot)$, mentioned in the Lemma's statement is denoted by $x^\circ(t)$ and is termed standard.

1. We prove relation (4.1). Assume that the first player uses Rule 1. Let $x_0 = x(t_0) \in C \setminus [ma)$. By $t[x_0, v(\cdot)]$ we denote the first instant that system (1.1) hits onto curve ma. We fix an arbitrary program $v(\cdot) \in V[x_0|C]$. The motion $x(t)$ of system (1.1) by virtue of this program and of Rule 1 is called a phase motion (to distinguish it from the standard motion). We set $t[x_0] = \max\{t[x_0, v(\cdot)], t[x_0, v^*(\cdot)]\}$. Let us show that at the instant $t[x_0]$ the point $x(t[x_0])$ is located on the curve ma not farther in relation to point m than the point $x^\circ(t[x_0])$ (notation: $x(t[x_0]) \leqslant x^\circ(t[x_0])$). The validity of relation (4.1) for any $x_0 \in C$ follows from the obvious validity of this relation for $x_0 \in (ma)$ as a corollary of the last assertion.

Through the points h_1, h_4 we draw a straight line P_3R_3 (Fig. 3). Let $C^{(1)}$ $(C^{(2)})$ be the part of set C lying above (below) this straight line. The intersection of the straight line P_3R_3 with set C is included in $C^{(1)}$. Assume that $x_0 \in C^{(1)}$. The standard motion on the interval $[t_0, t[x_0, v^*(\cdot)])$ will be attracted to the point h_4, while at any instant $t \in [t_0, t[x_0, v(\cdot)])$ the phase motion will be attracted to some point (depending on t) of the segment $[h_1, h_4]$. Hence, with due regard to the nature of the orientation of segment $[h_1, h_4]$ we get that when $t \geqslant t_0$ both motions will pass into $C^{(1)}$ and the points where they hit onto the curve ma are connected by the relation $x(t[x_0, v(\cdot)]) \leqslant x^\circ(t[x_0, v^*(\cdot)])$. Consequently, to prove our assertion we need only examine the case $t[x_0, v^*(\cdot)] < t[x_0, v(\cdot)]$. It is possible only when the curve ma decreases monotonically with respect to x_1. In this case, for any $t \in [t[x_0, v^*(\cdot)], t[x_0])$ the standard motion is attracted to some point of the segment $[h_3, h_4]$, whereas the phase motion is attracted to a point of the segment $[h_1, h_4]$. Therefore, for any $t \in [t_0, t[x_0])$ we have $x_1^{\cdot}(t) \leqslant x_1^{0\cdot}(t)$, and, hence, $x_1(t[x_0]) \leqslant x_1^{\circ}(t[x_0])$. Since at the instant $t[x_0]$ both motions are found to be on the curve ma, (keeping in mind the nature of this curve) we obtain: $x(t[x_0]) \leqslant x^{\circ}(t[x_0])$. By means of arguments of the very same kind we can prove the relation $x(t[x_0]) \leqslant$ $\leqslant x^{\circ}(t[x_0])$ also for $x_0 \in C^{(2)}$.

2. Let $x_0 \in C$ and $v = -\nu$. The program $u_*(\cdot) \in U[x_0, v^*(\cdot)|A]$, solving the problem of transferring system (1.1) in least time from the point x_0 to the point m without its leaving set A, has the following structure: it equals $-\mu$ upto the instant that system (1.1) hits onto the curve ma, and subsequently it effects the motion of system (1.1) along the curve ma upto the instant of hitting onto point m. This fact follows from a qualitative analysis of the reachable region [1] of system (2.2) from the point $x(\tau_0) = m$ with $v = -\nu$ and under the phase constraint $x(\tau) \in C$, $\tau \geqslant \tau_0$. Obviously, $u_*(\cdot) = u^*(\cdot)$. Hence the lemma's assertion follows from relation (4.1). The Lemma is proved.

For any $x_0 \in A_1 \subset C$ the set $V[x_0 | C]$, coincides, as is not difficult to verify, with the set of all possible programs $v(\cdot)$. Therefore, in the set A_1 the time $T_1[x_0] =$ $= T^{(1)}[x_0]$. From among the curves of family L let us try to find a curve df° which separates out from the set A a maximal closed subset C° (containing A_1), for each point x_0 of which the equality $T_1[x_0] = T^{(1)}[x_0]$ is satisfied. Such a curve exists. We can indicate a method for constructing a sequence of curves whose limit it is. We restrict ourselves to listing some properties of this curve.

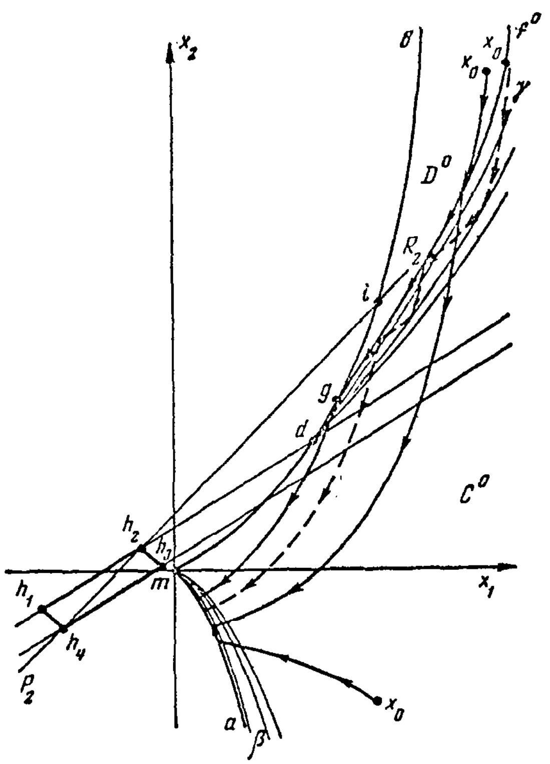

Fig. 5

By i we denote the point of intersection of the straight line P_2R_2 with curve db (Fig. 5). If there is no intersection we take it that the point i lies at infinity (on the curve db)

1. The curve df° is a smooth trajectory of the motion of system (2.2) from point d with $v = \nu$.

2. The relation $r^{(1)}(x_*, -\mu) <$ $< x_* f^{\circ} < r^{(2)}(x_*, -\mu)$ is valid for any point x_* on curve df°

Let $x^{(1)}, x^{(2)}$ be arbitrary points on curve df° and let $x^{(1)}$ be located closer to d as compared to $x^{(2)}$.

3. The maximum $T^{(3)}[x^{(2)}, x^{(1)}]$ of the time $T^{(3)}[x^{(2)}, x^{(1)}; v(\cdot)]$ it takes system (1.1) to move from the point $x^{(2)}$ to the point $x^{(1)}$ along curve df° taken over all programs $v(\cdot)$ for which such a motion is possible, is reached on the program $v(t) \equiv \nu$.

4. The inequality

$$T^{(3)}\ [x_0,\ d] + T_1\ [d] \leqslant T_1\ [x_0] \tag{4.2}$$

is valid for any point x_0 on curve df°

5. Let g be the point on curve df°, for which the equality sign in (4.2) is first achieved when we go from the point d. Then, the equality sign is achieved in (4.2) for any point x_0 on the curve gf°. The arc dg of curve df° lies completely on the arc di of curve db.

6. If point $x^{(1)}$ belongs to the curve $[dg)$ ($[gf^\circ)$) then the inequality (equality)

$$T^{(3)}\,[x^{(2)}, x^{(1)}] + T_1\,[x^{(1)}] < T_1\,[x^{(2)}] \qquad (T^{(3)}\,[x^{(2)}, x^{(1)}] + T_1\,[x^{(1)}] = T_1\,[x^{(2)}])$$

is valid.

The curve gf° – a part of the curve df°_{+} – is called an equivocal curve [2]. We pose two auxiliary problems.

Probelm 4.2 (4.3). Find right-piecewise-continuous programs

$$v^*(\cdot),\ u^*(\cdot) \in U\,[x_0,\ v^*(\cdot)\,|\,A]\,(u^*(\cdot) \in U\,[x_0,\ v^*(\cdot)\,|\,D^\circ],\ D^\circ = A \setminus C^\circ)$$

satisfying the relation

$$\max\nolimits_{v(\cdot)} \min\nolimits_{u(\cdot)} T_1\,[x_0; u(\cdot), v(\cdot)] = T_1\,[x_0; u^*(\cdot), v^*(\cdot)], \qquad x_0 \in C^\circ \tag{4.3}$$

$$(T_2\,[x_0] = \max\nolimits_{v(\cdot)} \min\nolimits_{u(\cdot)} \{T_2\,[x_0; u(\cdot), v(\cdot)] + T_1\,[x\,[x_0; u(\cdot), v(\cdot)]]\} = = T_2\,[x_0; u^*(\cdot), v^*(\cdot)] + T_1\,[x\,[x_0; u^*(\cdot), v^*(\cdot)]], x_0 \in D^\circ)$$

where the maximum is taken over all possible programs $v(\cdot)$ and the minimum over all programs

$$u(\cdot) \in U\,[x_0,\ v(\cdot)\,|\,A] \qquad (u(\cdot) \in U\,[x_0,\ v(\cdot)|\,D^\circ])$$

and $x\,[x_0;\ u(\cdot),\ v(\cdot)]$ is the notation for the point of first entry of system (1.1) onto the curve df° from the point $x_0 \in D^\circ$ by virtue of programs $u(\cdot), v(\cdot)$.

The solution of Problem 4.2 ensues from the solution of Problem 4.1, from the properties of curve df° and from the equality $T_1\ [x_0] = T^{(1)}\ [x_0]$, $x_0 \in C^\circ$. When $x_0 \in C^\circ \setminus (gf^\circ)$ the program $v^*(\cdot)$ is unique: $v^*(t) \equiv -\nu$. For points x_0 on the curve (gf°) the maximum in (4.3) is reached on any program $v^*(\cdot)$ of the form

$$v^*(t) = \begin{cases} \nu, & \text{if} \quad t_0 \leqslant t < t_0 + \Delta t \\ -\nu, & \text{if} \quad t \geqslant t_0 + \Delta t \end{cases} \qquad 0 \leqslant \Delta t \leqslant T^{(3)} \quad [x_0, g]$$

In any case the program $u^*(\cdot)$ coincides with the realization $u(\cdot)$ formed in accordance with Rule 1 with $v(t) = v^*(t)$. The solution of Problem 4.3 is described in Lemma 4.2, its proof is omitted.

Lemma 4.2. For any initial position $x_0 \in D^\circ$ the solution of Problem 4.3 is unique. For any $x_0 \in D^\circ$ we have $v^*(t) \equiv \nu$, and the program $u^*(\cdot)$ coincides with the realization $u(\cdot)$ formed in accordance with Rule 1 with $v(t) \equiv \nu$. The program $v^*(\cdot)$ satisfies the relation

$$T_2\,[x_0; u^*(\cdot), v^*(\cdot)] + T_1\,[x\,[x_0; u^*(\cdot), v^*(\cdot)]] = = \max\nolimits_{v(\cdot)} \{T_2\,[x_0; u^*(\cdot), v(\cdot)] + T_1\,[x\,[x_0; u^*(\cdot), v(\cdot)]]\} \qquad (x_0 \in D^\circ)$$

where the maximum is taken over all possible programs $v(\cdot)$.

Typical trajectories of the motion of system (1.1) in the set C° (D°) by virtue of programs $v^*(\cdot)$, $u^*(\cdot)$, solving Problem 4.2 (4.3), are shown on Fig. 5. Thus, Rule 1 guarantees the first player the time

$$T^{(1)}[x_0] = \begin{cases} T_1[x_0], & \text{if} \quad x_0 \in C^\circ \\ T_2[x_0], & \text{if} \quad x_0 \in D^\circ \end{cases} \tag{4.4}$$

It is not difficult to establish that the function $T^{(1)}[x_0]$ is continuous in the set $A = C^\circ \cup D^\circ$. In the set $A \setminus \{m\}$ let us construct, on the basis of Rule 1, the optimal tactic $\{u^\circ, \delta^\circ\}$. We set

$$u^\circ[x] = \begin{cases} -\mu, & \text{if} \quad x \in C^\circ \setminus ((ma) \cup (df^\circ)) \\ \mu, & \text{if} \quad x \in D^\circ \cup (ma) \cup (df^\circ) \end{cases}$$

The second element of the tactic – the sequence $(\delta_n^\circ[x])$ on the curve (ma) $((df^\circ))$ – is specified by means of a fan-shaped sequence (β_n) $((\gamma_n))$ of auxiliary curves converging to the curve ma (df°) as $n \to \infty$. Each curve of the sequence should leave the point m (d) and pass into the set $C^\circ \setminus (ma)$ $(C^\circ \setminus (df^\circ))$. (Several curves of the sequence (β_n) $((\gamma_n))$ are shown on Fig. 5 where they have been denoted by β (γ).)

At a point x_0 on the curve (ma) $((df^\circ))$ we set $\delta_n^\circ[x_0]$ equal to the least time, with respect to $v(\cdot)$ for system (1.1) to move from the point x_0 upto the curve β_n (γ_n) with $u = \mu$. At the remaining points of set $A \setminus \{m\}$ we take $\delta_n^\circ[x] = \delta^\circ[x]$ for any n. We choose the function $\delta^\circ[x]$ such that for any arbitrary initial position

$$x_0 = x(t_0) \in C^\circ \setminus ((ma) \cup (df^\circ)) \quad (x_0 = x(t_0) \in D^\circ), \qquad u = -\mu \quad (u = \mu)$$

system (1.1) with any realization $v(\cdot)$ may not leave the set C° (D°) in the interval $[t_0, t_0 + \delta^\circ[x_0]]$. The tactic $\{u^\circ, \delta^\circ\}$ described ensures the "sliding" of system (1.1) along the curve ma in the direction of point m for any realization $v(\cdot)$. Here the motion does not go outside of set A. The appearance of a sliding mode on the curve df° is already connected with a concrete form of the realization $v(\cdot)$. A possible trajectory of the motion of system (1.1) when the first player uses the functions $u^\circ[x]$, $\delta_n^\circ[x]$ is shown by a dotted line in Fig. 5.

Let us sketch the proof of the optimality of tactic $\{u^\circ, \delta^\circ\}$. Let x_0 be an arbitrary initial position in set A. We fix an arbitrary program $v(\cdot)$. As $n \to \infty$ the sequence of trajectories of the motions of system (1.1) from the point x_0 by virtue of program $v(\cdot)$ and of functions $u^\circ[x]$, $\delta_n^\circ[x]$ converges to the trajectory of the motion of system (1.1) from the point x_0 by virtue of program $v(\cdot)$ and of Rule 1. Here the sequence $(T[x_0; u^\circ, \delta_n^\circ, v(\cdot)])$ converges to the quantity $T^{(1)}[x_0, v(\cdot)]$. Consequently,

$$\sup_{v(\cdot)} T^{(1)}[x_0, v(\cdot)] = \sup_{v(\cdot)} \overline{\lim_{n\to\infty}} T[x_0; u^\circ, \delta_n^\circ, v(\cdot)], \qquad x_0 \in A \tag{4.5}$$

Here the least upper bound is taken over all possible programs $v(\cdot)$. Further, for a given concrete tactic $\{u^\circ, \delta^\circ\}$ we can prove the equality

$$\sup_{v(\cdot)} \overline{\lim_{n\to\infty}} T[x_0; u^\circ, \delta_n^\circ, v(\cdot)] = \overline{\lim_{n\to\infty}} \sup_{v(\cdot)} T[x_0; u^\circ, \delta_n^\circ, v(\cdot)], \qquad x_0 \in A \tag{4.6}$$

where too the least upper bound is taken over all possible programs $v(\cdot)$. It is obvious that the least upper bound of the quantity

$$T[x_0; u^\circ, \delta_n^\circ, v(\cdot)] \quad (T^{(1)}[x_0, v(\cdot)]), \qquad x_0 \in A$$

over all possible programs $v(\cdot)$ coincides with the least upper bound of this quantity over all possible realizations $v(\cdot)$. With due regard to this fact, the optimality of the tactic $\{u^\circ, \delta^\circ\}$ ensues from the equalities (4.5), (4.6) and from the meaning of the time $T^{(1)}[x_0]$ (see the definition of this time and see formula (4.4)).

The known form of the maximin programs $v^*(\cdot)$ solving Problems 4.2, 4.3 right away permits us to determine, keeping inequality (1.3) in mind, the optimal function $v^\circ[x]$

$$v^\circ[x] = \begin{cases} -\nu, & \text{if} \quad x \in C^\circ \setminus (gf^\circ) \\ \nu, & \text{if} \quad x \in D^\circ \\ \text{either } \nu, \text{ or } -\nu, & \text{if} \quad x \in (gf^\circ) \end{cases}$$

Thus, for any initial position $x_0 \in A$ the tactic $\{u^\circ, \delta^\circ\}$ guarantees the first player the time

$$T_u[x_0] = T_v[x_0] = T^{(1)}[x_0] < \infty$$

while the function $v^\circ[x]$ does the same for the second player.

The limit (as $n \to \infty$) motion of system (1.1) from a point $x_0 \in A$ by virtue of functions $u^\circ[x]$, $\delta_n^\circ[x]$ and of functions $v^\circ[x]$ with a discrete step Δ_n is called a standard motion. By looking over all possible methods of specifying the function $v^\circ[x]$ on the equivocal curve gf°, we obtain a complete collection of standard motions issuing from the one point x_0 and reaching along different trajectories the point m in one and the same length of time. When $x_0 \in C^\circ$ this collection is contained in the set of all motions of system (1.1) from point x_0 by virtue of the programs $v^*(\cdot)$, $u^*(\cdot)$, solving Problem 4.2. When $x_0 \in D^\circ$ the standard motion coincides, upto the instant of hitting onto the curve gf° with the motion of system (1.1) by virtue of the programs $u^*(\cdot)$, $v^*(\cdot)$, solving Problem 4.3.

The analysis of Case 2.1 has been completed. We dwell briefly on the remaining cases. The single difference between case 2.2 and the one just analyzed is that the time $T_u[x_0] = T_v[x_0] = \infty$ on the curve (db) (in this case it is a halfline). In case 2.2 the point g coincides with point d.

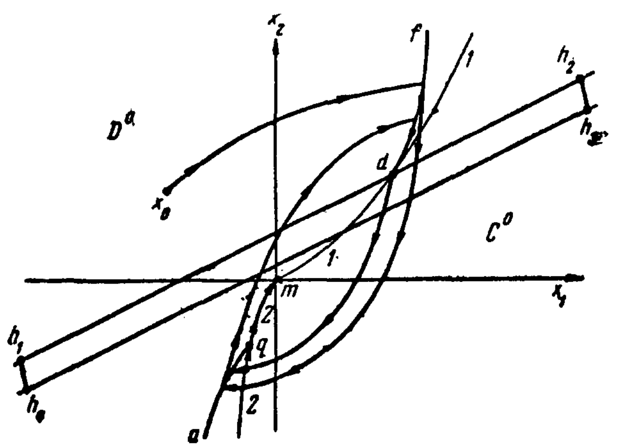

Fig. 6

In cases 3.3, 3,4 the standard motion goes along the curve me before hitting the point m. The solution is similar to the solution in case 2.1: the set A is divided in the same way into two sets C° and D°, and in them the tactic $\{u^\circ, \delta^\circ\}$ and the function $v^\circ[x]$ are determined in the same way. In the set A the time $T_u[x_0] = T_v[x_0] < \infty$ and has the same meaning as before.

Cases 1.1, 1.2 (3.1, 3.2) can be looked upon as a degeneration of case 2.1(3.3) when $A = A_1$. In these cases the equivocal curves do not arise.

The solution in Case 2.3 is most complicated in nature (Fig. 6). Let A_1 be a closed curvilinear cone containing the x_1-semiaxis and bounded by the curves $r^{(1)}(m, \mu)$ and $r^{(2)}(m, -\mu)$. A certain curve $aqmdf$ divides the set $A = X$ into two parts $C^\circ \supset A_1$ and $D^\circ = A \setminus C^\circ$; the curve $aqmdf$ is included in C°. The arcs md and qm

of the curve $aqmdf$ abutting point m are arcs of the curves $r^{(2)}(m, -\mu)$ and $r^{(1)}(m, \mu)$ respectively. The curves df and aq are equivocal, they pass into the set $A \setminus A_1$. The location of point q on the curve $r^{(1)}(m, \mu)$ depends upon the distance between the point d and h_2. The functions $u^\circ[x]$ and $v^\circ[x]$ are determined in the following way:

$$u^\circ[x] = \begin{cases} -\mu, & \text{if } x \in C^\circ \setminus ([mqa) \cup (df)) \\ \mu, & \text{if } x \in D^\circ \cup (mqa) \cup (df) \end{cases}$$

$$v^\circ[x] = \begin{cases} -\nu, & \text{if } x \in C^\circ \setminus ((qa) \cup (df)) \\ \nu, & \text{if } x \in D^\circ \\ \text{either } \nu, \text{ or } -\nu, & \text{if } x \in (qa) \cup (df) \end{cases}$$

The sequence $(\delta_n^\circ[x])$ is given with the aid of auxiliary curves. The time $T_u[x_0] = T_v[x_0] < \infty$ on the whole plane. The function $T_u[x_0]$ undergoes a discontinuity on the curve $[qmd]$. It is continuous in the rest of the plane.

Figures 2, 4, 6 show typical standard trajectories for cases 1.1, 3.3, 2.3, respectively. Thus, the solutions of Problems 1 and 2 for the cases not covered in the hypotheses of Lemmas 2.1, 2.2 have been found completely. Namely, a partitioning of the plane into two sets A and B has been indicated. For $x_0 \in A$ the time $T_u[x_0] = T_v[x_0] < \infty$ (the only exception is case 2.2 in which the time $T_u[x_0] = T_v[x_0] = \infty$ on the halfline (db) lying on the boundary of A). For $x_0 \in B$ the time $T_u[x_0] = T_v[x_0] = \infty$. The first player's optimal tactic $\{u^\circ, \delta^\circ\}$ and the second player's optimal function $v^\circ[x]$ have been determined in set A. The optimal function $v^\bullet[x]$ has been found in set B.

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